LESSON 16 - STUDY GUIDE

Abstract. In this lesson we will look at Fourier series and their convergence in norm for functions in $L^2(\mathbb{T})$. The results are direct applications of some of the fundamental theorems in the theory of Hilbert spaces, and therefore not extendable to the other $L^p(\mathbb{T})$ spaces, which makes the $L^2(\mathbb{T})$ case very special, easier and significantly more impeccable.

1. Fourier series in $L^2(\mathbb{T})$.

Study material: We will follow closely the section **5 - Fourier Series of Square Summable Functions** from chapter **I - Fourier Series on** $\mathbb{T}$, corresponding to pgs. 27–30 in the second edition [2] and pgs. 29–32 in the third edition [3] of Katznelson's book. Most of the material is just the basic theory of Hilbert spaces, for which a more complete presentation can be easily reviewed from most Functional Analysis books, and I suggest in particular section **5.5 - Hilbert Spaces** from chapter **5 - Elements of Functional Analysis** in Folland's book [1].

In this lesson we will study the convergence of the partial sums of Fourier series in the $L^2(\mathbb{T})$ norm. It is the first example of an L^p norm convergence and we will see how elegant the results in this space are. This is, of course, a consequence of the Hilbert space structure of $L^2(\mathbb{T})$. No other L^p is a Hilbert space, and in fact L^2 is the one of the most suitable contexts - jointly, in $\mathbb{R}^n$, with the Schwartz space $\mathcal{S}$ of rapidly decreasing functions and the tempered distributions - for doing Fourier analysis, as we will see below. In $L^2(\mathbb{T})$ one can basically look at Fourier series of functions as expansions in an orthonormal basis, thus generalizing the analogous concepts of Euclidean linear algebra to infinite dimensions. This point of view requires an inner-product space, of course, and therefore cannot be transposed to the other $L^p(\mathbb{T})$ spaces, which are only Banach spaces whose norms do not arise from inner-products.

Most of the results about the Fourier transform and Fourier series in $L^2(\mathbb{T})$ are straightforward applications of the theory of Hilbert spaces, which is assumed to be known as a prerequisite for the current course, so we will not cover all the details (if necessary, see the Section 5.5 - Hilbert Spaces, in Chapter 5 of Rolland's Real Analysis book [1] for a fast and complete overview of the fundamental theory).

Following Katznelson [2, 3], we will just review a few properties of inner-product and Hilbert spaces, from which the Fourier series results in $L^2(\mathbb{T})$ readily follow. The first basic property to keep in mind is that Pythagoras' theorem still holds in these spaces.

Theorem 1.1. (Pythagoras Theorem) *Let $\{\phi_n\}_{1\le n\le N}$, be a finite orthonormal set of elements in an inner-product linear space, i.e. such that $\langle \phi_i, \phi_j\rangle = 0$ when $i \neq j$, and $\langle \phi_i, \phi_i\rangle = 1$. Then, for $a_1, a_2, \ldots, a_N \in \mathbb{C}$,*

$$\left\| \sum_{n=1}^{N} a_n \phi_n \right\|^2 = \sum_{n=1}^{N} |a_n|^2.$$

The proof is simply an exercise in computing the square of the norm on the left hand side of the identity from the inner-product of the sum with itself, and using the orthonormality relations of the functions ϕ_n.

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An immediate consequence of the Pythagorean theorem is that the projection of any vector onto the subspace generated by a finite set of orthonormal vectors will always have smaller norm than the vector itself. Unless, of course, the vector is actually in that subspace.

Proposition 1.2. *Let $\{\phi_n\}_{1\leq n\leq N}$, be a finite orthonormal set of elements in an inner-product linear space. Then*

$$\sum_{n=1}^{N} |\langle v, \phi_n\rangle|^2 \leq \|v\|^2,$$

for any vector v, with the equality being attained if and only if

$$v = \sum_{n=1}^{N} \langle v, \phi_n\rangle \phi_n.$$

Proof. For any vector v we have

$$\|v - \sum_{n=1}^{N} \langle v, \phi_n\rangle \phi_n\|^2 \geq 0,$$

and expanding the square of the norm

$$\|v - \sum_{n=1}^{N} \langle v, \phi_n\rangle \phi_n\|^2 = \left\langle v - \sum_{n=1}^{N} \langle v, \phi_n\rangle \phi_n, v - \sum_{n=1}^{N} \langle v, \phi_n\rangle \phi_n \right\rangle = \langle v, v\rangle - \sum_{n=1}^{N} |\langle v, \phi_n\rangle|^2, \tag{1.1}$$

from which the conclusion follows. □

So, from this proposition, we conclude that the norm of a vector is an upper bound for the norms of the projections onto subspaces generated by whatever finite sets of orthonormal $\{\phi_n\}_{1\leq n\leq N}$ we pick. So, if we take any set of orthonormal vectors whatsoever, $\{\phi_\alpha\}_{\alpha\in A}$, finite or infinite, countable or uncountable, then this proposition still yields

$$\sup_{F\subset A} \sum_{\alpha\in F} |\langle v, \phi_\alpha\rangle|^2 \leq \|v\|^2,$$

where the subsets of indices $F \subset A$ are finite. If we now define the sum of nonnegative numbers $\{x_\alpha\}_{\alpha\in A}$, $x_\alpha \geq 0$, over any set of indices A, as

$$\sum_{\alpha\in A} x_\alpha = \sup_{F\subset A} \sum_{\alpha\in F} x_\alpha, \tag{1.2}$$

and observe that this coincides with the integral over A with the counting measure, as well as with the usual sum of series, when $A = \mathbb{N}$, then the following important conclusion becomes a direct consequence of Proposition 1.2.

Corollary 1.3. (Bessel's Inequality) *Let $\{\phi_\alpha\}_{\alpha\in A}$ be any - finite or infinite - set of orthonormal vectors of an inner-product space. Then, for any vector v we have*

$$\sum_{\alpha\in A} |\langle v, \phi_\alpha\rangle|^2 \leq \|v\|^2.$$

It is a simple exercise to show that, if a sum of infinite terms as (1.2) is finite, then there can be only at most a countable number of x_α which are nonzero. So, from Bessel's inequality we conclude that, even if A is uncountable, there will be no more than a countable number of indices $\alpha \in A$ for which $\langle v, \phi_\alpha\rangle \neq 0$, for every vector v in the space.

If we now add the completeness ingredient to the inner-product space, and start looking at Hilbert spaces, the results become deeper and more far-reaching. To begin with, we can extend the Pythagorean identity to the infinite dimensional case.

Proposition 1.4. *Let H be a Hilbert space, and $\{\phi_n\}_{n=1}^{\infty}$ a sequence of orthonormal elements. If $\{a_n\}_{n=1}^{\infty}$ is a sequence of complex numbers, then the sequence of partial sums $\sum_{n=1}^{N} a_n \phi_n$ converges in the norm of H, i.e. the series $\sum_{n=1}^{\infty} a_n\phi_n$ converges in H, if and only if $\sum_{n=1}^{\infty} |a_n|^2 < \infty$. In that case, denoting by $v \in H$ the element to which the series converges, the infinite version of the Pythagorean identity holds*

$$\|v\|^2 = \left\| \sum_{n=1}^{\infty} a_n \phi_n \right\|^2 = \sum_{n=1}^{\infty} |a_n|^2, \tag{1.3}$$

and

$$a_n = \langle v, \phi_n \rangle, \quad \textit{for all} \quad n.$$

Proof. We start by proving that $\sum_{n=1}^{\infty} |a_n|^2 < \infty$ implies the convergence of the series $\sum_{n=1}^{\infty} a_n\phi_n$ in H. The argument is just like the one used earlier in the course, in the proof of convergence of absolutely convergent series in Banach spaces, except that here it uses the Pythagorean theorem instead.

So, if we take the difference of any two elements in the sequence of partial sums, say with $N > M$, then

$$\left\| \sum_{n=1}^{N} a_n\phi_n - \sum_{n=1}^{M} a_n\phi_n \right\|^2 = \left\| \sum_{n=M+1}^{N} a_n\phi_n \right\|^2 = \sum_{n=M+1}^{N} |a_n|^2,$$

from the Pythagorean identity. And because the series $\sum_{n=1}^{\infty} |a_n|^2$ converges, we can make the right hand side of this identity as small as we want by taking N, M large enough. And this proves that the sequence of partial sums $\sum_{n=1}^{N} a_n\phi_n$ is a Cauchy sequence in H which, being complete, implies that the partial sums converge in the norm to an element $v \in H$.

Identity (1.3) is obtained by simply using again the Pythagorean theorem for the partial sums

$$\left\| \sum_{n=1}^{N} a_n\phi_n \right\|^2 = \sum_{n=1}^{N} |a_n|^2,$$

and then taking the limit as $N \to \infty$, which, conversely, also establishes that this value is finite when the series converges.

The final identity is a consequence of the continuity of the inner product, by taking the limit

$$a_n = \left\langle \sum_{j=1}^{N} a_j\phi_j, \phi_n \right\rangle, \quad \text{for} \quad N \geq n,$$

when $N \to \infty$. □

Therefore, in Hilbert spaces, convergent series expansions in countable orthonormal sets of vectors are equivalent to the corresponding sequence of the coefficients being in l^2. It is also a simple exercise to prove that the convergence of the series is invariant under permutations of the order of summation.

The only ingredient left in order to make sure that any vector in a Hilbert space can be represented by a countable series expansion in an orthonormal set is to know whether that orthonormal set is complete, in the sense of there not being any orthonormal vectors left out that could still correspond to further independent directions that are not being considered. Equivalently, by looking at Bessel's inequality, the insufficiency of the sum on the left hand side, to turn the inequality into an identity, corresponds to an incompleteness of the orthonormal set to encompass all possible orthonormal directions, so that when the inequality is strict it implies that there is still a shortfall in orthogonal vectors. The idea is akin to checking if a basis generates the whole space, or just a subspace with some independent directions still missing. A set of orthonormal vectors in a Hilbert space is therefore defined as complete if no further independent orthonormal directions are being left out. Which is the same as the following.

Definition 1.5. *In a Hilbert space H a* complete orthonormal system*, also called a* Hilbert basis*, is an orthonormal set of vectors $\{\phi_\alpha\}_{\alpha\in A}$ in H such that, if $\langle v, \phi_\alpha\rangle = 0$ for all $\alpha \in A$ then $v = 0$.*

We thus have the main theorem.

Theorem 1.6. (General Riesz-Fischer Theorem)[1] *Let H be a Hilbert space and $\{\phi_\alpha\}_{\alpha\in A}$ a set of orthonormal vector. Then, the following are equivalent.*

(1) *$\{\phi_\alpha\}_{\alpha\in A}$ is complete*
(2) *For every $v \in H$ we have $v = \sum_{\alpha\in A}\langle v, \phi_\alpha\rangle\phi_\alpha$ where the sum has only at the most a countable number of nonzero terms and converges in the norm, no matter what the order of the terms is.*
(3) **(Plancherel's Identity)** *For every $v \in H$ we have $\|v\|^2 = \sum_{\alpha\in A} |\langle v, \phi_\alpha\rangle|^2$.*
(4) **(Parseval's Identity)** *For every $v, w \in H$ we have $\langle v, w\rangle = \sum_{\alpha\in A}\langle v, \phi_\alpha\rangle\overline{\langle w, \phi_\alpha\rangle}$.*

Proof. (1) $\Rightarrow$ (2): From Bessel's inequality we know that $\|v\|^2 \geq \sum_{\alpha\in A} |\langle v, \phi_\alpha\rangle|^2$, therefore that only a countable number of $\langle v, \phi_\alpha\rangle$ are nonzero and thus, from the previous proposition, that the series $\sum_{\alpha\in A}\langle v, \phi_\alpha\rangle\phi_\alpha$ converges, independently of the ordering of the terms, to some vector, say $\tilde{v} \in H$. But then, taking the vector $v - \tilde{v} = v - \sum_{\alpha\in A}\langle v, \phi_\alpha\rangle\phi_\alpha$ we have $\langle v - \tilde{v}, \phi_\alpha\rangle = \langle v, \phi_\alpha\rangle - \langle\tilde{v}, \phi_\alpha\rangle = \langle v, \phi_\alpha\rangle - \langle v, \phi_\alpha\rangle = 0$, for all $\alpha \in A$. And because the set $\{\phi_\alpha\}_{\alpha\in A}$ is complete, we conclude that $v - \tilde{v} = 0$, i.e. that $v = \tilde{v} = \sum_{\alpha\in A}\langle v, \phi_\alpha\rangle\phi_\alpha$.

(2) $\Rightarrow$ (1): If $v \in H$ is such that $\langle v, \phi_\alpha\rangle = 0$ for all $\alpha \in A$ and the series converges in norm to v, then $v = 0$ and this implies that $\{\phi_\alpha\}_{\alpha\in A}$ is complete.

(2) $\Rightarrow$ (3): Follows from the previous proposition.

(3) $\Rightarrow$ (2): We saw in (1.1) that, for a finite partial sum we have $\|v - \sum_{n=1}^N\langle v, \phi_n\rangle\phi_n\|^2 = \langle v, v\rangle - \sum_{n=1}^N |\langle v, \phi_n\rangle|^2$ and so, taking $N \to \infty$ in the countable sum of nonzero terms, if Plancherel's identity holds, then the series converges in the norm to v.

(2) $\Rightarrow$ (4): If $v = \sum_{\alpha\in A}\langle v, \phi_\alpha\rangle\phi_\alpha$ then, picking some ordering of the countable nonzero terms, we have, from the continuity of the inner product, $\langle v, w\rangle = \lim_{N\to\infty}\langle\sum_{n=1}^N\langle v, \phi_n\rangle\phi_n, w\rangle = \lim_{N\to\infty}\sum_{n=1}^N\langle v, \phi_n\rangle\langle\phi_n, w\rangle = \sum_{n=1}^\infty\langle v, \phi_n\rangle\overline{\langle w, \phi_n\rangle}$.

(4) $\Rightarrow$ (3): Obvious, making $w = v$. □

The final conclusion that this theorem therefore yields is that the possibility of expanding arbitrary vectors of a Hilbert space as a series summation of orthonormal set of vectors is equivalent to that set being complete, in the sense of the definition above. As this is totally analogous to Euclidean space expansions of vectors as linear combinations of orthonormal basis vectors, the terminology Hilbert basis is often also used, as mentioned in the definition of complete orthonormal system. But one must be careful because the word *basis* might be misleading here: a basis (also called Hamel basis) in finite, or infinite, dimensional linear algebra, is a set of linearly independent vectors such that any vector of the space can be written as a - necessarily unique - *finite* linear combination of elements of that set. That is not at all, in general, what happens for complete orthonormal systems in Hilbert spaces: although they are linearly independent, because they are pairwise orthogonal, the expansion of arbitrary vectors

[1] The naming of this family of results is not standard. Frigyes Riesz actually proved the specific result that functions in $L^2(\mathbb{T})$ could be represented by their Fourier series in norm, while Ernst Fischer proved the completeness of $L^2(\mathbb{T})$. But their names tend to be attached to the corresponding result in the general setting of any Hilbert space, as we do here, although it is often also associated to the theorem of completeness of L^p spaces. Plancherel proved that the L^2 norm of a function and of its Fourier transform are the same, in the setting of $\mathbb{R}^n$, but his name is more generally associated, in abstract harmonic analysis, with the identity between the L^2 norms of a function, on its locally compact abelian group, and of its Fourier transform, on the Pontryagin dual group. And consequently, the isometry of the corresponding L^2 spaces with their Haar measures. Finally, Parseval's name is commonly associated to the specific Hilbert space isomorphism between $L^2(\mathbb{T})$ and $l^2(\mathbb{Z})$.

in terms of them is done generally through convergent infinite series. So a Hilbert basis generally is not a linear algebra (Hamel) basis.

But pretty much in the same way as one proves, using Zorn's lemma, that any vector space has a (Hamel) basis, it can also be proved that any Hilbert space has a complete orthonormal system, or Hilbert basis.

Proposition 1.7. *Every Hilbert space has a complete orthonormal system.*

Another crucial result has to do with the cardinality of complete orthonormal systems. It turns out that separability, i.e. the existence of a countable dense subset of the Hilbert space, is equivalent to any Hilbert basis being countable too.

Proposition 1.8. *A Hilbert space has a countable complete orthonormal system if and only if it is separable. In this case, all such complete systems are countable.*

The proofs of these propositions can be found in Folland's book [1], in Section 5.5, on Hilbert spaces.

We can finally apply these results to $L^2(\mathbb{T})$, which is a Hilbert space with inner-product given by

$$\langle f, g\rangle = \frac{1}{2\pi}\int_{\mathbb{T}} f(t)\overline{g(t)}dtdt,$$

from which the $L^2(\mathbb{T})$ norm follows as

$$\|f\|_{L^2(\mathbb{T})} = \left(\frac{1}{2\pi}\int_{\mathbb{T}} |f(t)|^2 dt\right)^{\frac{1}{2}} = \sqrt{\langle f, f\rangle},$$

and the completeness in this norm is the one that we proved at the beginning of the course, for all the L^p spaces.

It is a simple exercise in the theory of L^p spaces to prove that all of them are separable, except for L^∞. Therefore $L^2(\mathbb{T})$ has a countable orthonormal complete system. Obviously, one such system is going to be the set of exponentials $\{e^{int}\}_{n\in\mathbb{Z}}$, of the Fourier series. In any case, we can simply prove its completeness directly, from which separabilty will then follow.

Proposition 1.9. *The set of functions in $L^2(\mathbb{T})$, $\{e^{int}\}_{n\in\mathbb{Z}}$, is a complete orthonormal system.*

Proof. That the set of exponentials is orthonormal is the same as the repeatedly used property that

$$\langle e^{int}, e^{imt}\rangle = \frac{1}{2\pi}\int_{\mathbb{T}} e^{int}e^{-imt}dt = \begin{cases} 0 & \text{if} \quad n \neq m \\ 1 & \text{if} \quad n = m \end{cases}.$$

Their completeness, on the other hand, is a direct consequence of the uniqueness of Fourier coefficients seen in the previous lesson. For, if $f \in L^2(\mathbb{T}) \subset L^1(\mathbb{T})$ is such that

$$\langle f, e^{int}\rangle = \frac{1}{2\pi}\int_{\mathbb{T}} f(t)e^{-int}dt = \hat{f}(n) = 0,$$

for all $n \in \mathbb{Z}$, then we know that f is the zero function. And this completes the proof. □

Corollary 1.10. *$L^2(\mathbb{T})$ is a separable space, i.e. it has a countable dense subset.*

Therefore, we can now directly apply Theorem 1.6 to obtain the following fundamental properties of Fourier series in $L^2(\mathbb{T})$.

Theorem 1.11. (Riesz-Fischer Theorem for $L^2(\mathbb{T})$) *Let $f, g \in L^2(\mathbb{T})$. Then*

(1) $f = \displaystyle\sum_{n=-\infty}^{\infty} \langle f, e^{int}\rangle e^{int} = \lim_{N\to\infty}\sum_{n=-N}^{N} \hat{f}(n)e^{int}$ *in the $L^2(\mathbb{T})$ norm.*

(2) **(Plancherel's Identity)** $\|f\|^2_{L^2(\mathbb{T})} = \dfrac{1}{2\pi}\displaystyle\int_{\mathbb{T}} |f(t)|^2 dt = \sum_{n=-\infty}^{\infty} |\hat{f}(n)|^2.$

(3) **(Parseval's Identity)** $\langle f, g\rangle = \dfrac{1}{2\pi}\displaystyle\int_{\mathbb{T}} f(t)\overline{g(t)} dt = \sum_{n=-\infty}^{\infty} \hat{f}(n)\overline{\hat{g}(n)}.$

If we recall that the space $l^2(\mathbb{Z})$ of square summable complex sequences is also a Hilbert space, with inner product given by

$$\langle \{a_n\}, \{b_n\}\rangle_{l^2(\mathbb{Z})} = \sum_{n=-\infty}^{\infty} a_n \overline{b_n},$$

and corresponding norm

$$\|\{a_n\}\|^2_{l^2(\mathbb{Z})} = \langle \{a_n\}, \{a_n\}\rangle_{l^2(\mathbb{Z})} = \sum_{n=-\infty}^{\infty} |a_n|^2,$$

then what this whole Hilbert space theory applied to the $L^2(\mathbb{T})$ case has established is that the Fourier transform is a unitary bijective map, i.e. a Hilbert space isomorphism, between $L^2(\mathbb{T})$ and $l^2(\mathbb{Z})$ (note that Proposition 1.4 guarantees surjectivity). This is the reason why L^2 is unmatched, among all the L^p spaces, in terms of the success and perfection of the results obtained with the Fourier series.

We conclude this lesson with an important application of the Riesz-Thorin interpolation theorem to the Fourier transform. For, up until this point in our study of Fourier series, we have established that

$$\mathcal{F} : L^1(\mathbb{T}) \to l^\infty(\mathbb{Z}), \quad \text{with} \quad \|\mathcal{F}(f)\|_{l^\infty(\mathbb{Z})} \leq \|f\|_{L^1(\mathbb{T})},$$

and

$$\mathcal{F} : L^2(\mathbb{T}) \to l^2(\mathbb{Z}), \quad \text{with} \quad \|\mathcal{F}(f)\|_{l^2(\mathbb{Z})} = \|f\|_{L^2(\mathbb{T})}.$$

As the Fourier transform is a linear operator and $L^1(\mathbb{T}) + L^2(\mathbb{T}) = L^1(\mathbb{T}) \supset L^p(\mathbb{T})$, for all $1 \leq p \leq 2$, we can interpolate it between $L^1(\mathbb{T})$ and $L^2(\mathbb{T})$ to obtain the following bound.

Theorem 1.12. **(Hausdorff-Young Inequality)** *For $1 \leq p \leq 2$, the Fourier transform maps $L^p(\mathbb{T})$ to $l^q(\mathbb{Z})$, where $\frac{1}{p} + \frac{1}{q} = 1$, and we have*

$$\|\mathcal{F}(f)\|_{l^q(\mathbb{Z})} \leq \|f\|_{L^p(\mathbb{T})} \Leftrightarrow \left(\sum_{n=-\infty}^{\infty} |\hat{f}(n)|^q\right)^{\frac{1}{q}} \leq \left(\frac{1}{2\pi}\int_{\mathbb{T}} |f(t)|^p dt\right)^{\frac{1}{p}}.$$

Proof. Just apply Riesz-Thorin's interpolation theorem between the two cases, and check that, for $\theta \in [0, 1]$ such that

$$\frac{1}{p} = \frac{1-\theta}{1} + \frac{\theta}{2},$$

we get

$$\frac{1}{q} = \frac{1-\theta}{\infty} + \frac{\theta}{2},$$

and therefore $\frac{\theta}{2} = \frac{1}{q}$ which implies

$$\frac{1}{p} = \frac{1-\theta}{1} + \frac{\theta}{2} = 1 - \frac{\theta}{2} = 1 - \frac{1}{q},$$

and this proves that p and q are conjugate exponents. ☐

It is tempting to try to extrapolate this result for all $p > 2$ as well. Recalling from the theory of L^p spaces, covered at the beginning of the course, that, just like on the circle $\mathbb{T}$ one has the inclusions $L^\infty(\mathbb{T}) \subset L^p(\mathbb{T}) \subset L^1(\mathbb{T})$, for $\mathbb{Z}$ with the counting measure, one has the opposite inclusions $l^1(\mathbb{Z}) \subset l^q(\mathbb{Z}) \subset l^\infty(\mathbb{Z})$, then it does look as if the Fourier transform would continue the pattern of restricting the $l^q(Z)$ space of its range, from $q = \infty$ down to $q = 1$, as we restricted the $L^p(\mathbb{T})$ domain from $p = 1$ to $p = \infty$. As it happens, though, this is really only true between $L^1(\mathbb{T})$ and $L^2(\mathbb{T})$, with ranges between $l^\infty(\mathbb{Z})$ and $l^2(\mathbb{Z})$, from the Riesz-Thorin interpolation theorem as above. Because there exist continuous functions, therefore in all $L^p(\mathbb{T})$, for which $\sum_{-\infty}^{\infty} |\hat{f}(n)|^{2-\varepsilon} = \infty$, for any $\varepsilon > 0$. It is thus impossible to restrict the range of the Fourier transform to $l^q(\mathbb{Z})$, for $q < 2$, and the best that one can say when $f \in L^p(\mathbb{T}) \subset L^2(\mathbb{T})$, for $p > 2$, even for continuous functions, is that the Fourier transform is in $l^2(\mathbb{Z})$ and therefore that $\sum_{-\infty}^{\infty} |\hat{f}(n)|^2 < \infty$.

References

[1] Gerald B. Folland, *Real Analysis, Modern Techniques and Applications,* 2nd Edition, John Wiley & Sons, 1999.
[2] Yitzhak Katznelson *An Introduction to Harmonic Analysis,* 2nd Edition, Dover Publications, 1976.
[3] Yitzhak Katznelson *An Introduction to Harmonic Analysis,* 3rd Edition, Cambridge University Press, 2004.